Effective Dynamics

Jürg Fröhlich
ETH Zurich
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"Why should I blame anyone but myself if I cannot understand what I know nothing about?" – Pablo Picasso

Credits and Contents

Credits:

Abou Salem, Albanese, Bach, Ballesteros, Bauerschmidt, Benettin, Blanchard, De Roeck, Faupin, Fraas, Gang, Giorgilli, Griesemer, Knowles, Merkli, Pizzo, Schenker, Schlein, Schnelli, Schubnel, Schwarz, Sigal, Spencer, Ueltschi, Wayne. –

Today, two recent examples from quantum theory, with: *Schenker; Ballesteros, Fraas, Schubnel*

Contents:

1. Introduction – survey of examples
2. Thermal noise kills Anderson localization
3. Dynamics of a Quantum System under repeated measurements
4. Indirect measurements & "pointer observables"
5. Analogy to classical statistical mechanics
6. Effective dynamics of pointer observables
7. Conclusions

1. Introduction – survey of examples

- Return to equilibrium – Master equations, kinetic limit, etc. (examples: equilibration of spins or oscillators, thermal ionization, easy half of *0th Law*,…; with B & Sig; M – see also Jac & Pil)
- Relaxation to a ground state – scattering techniques, cluster expansions in real-time variable - DeR (example: decay of excited atom or molecule; preparation of states in QM, etc.; with Gr,S; Schub)
- Approach to a NESS or to a NE time-periodic state – scattering techniques, resonance theory for Liouvillians, dynamics with *entropy production,* Onsager relations (application: e.g., *2nd Law of Thermodynamics*; with M & Ue, A-S)
- Anderson Localization – **multi-scale analysis**, small denominators (examples: absence of diffusion for a quantum particle in a random- or 1D qp potential, absence of heat transport in disordered arrays of anharmonic oscillators, …; with Sp; Sp & W; A; B & G)

Survey of examples, ctd.

- Quantum Brownian Motion, position-space decoherence, equipartition – derivation of diffusive motion “from scratch” (example: atom with finitely many internal states coupled to a qm heat bath and hopping on a lattice; with DeR, Piz)
- Static disorder *and* thermal noise: Weak diffusive transport (examples: dyn. of a BEC, electron transport in a sparsely populated conduction band of a disordered semi-conductor: with J. Schenker)
- Quantum- and Hamiltonian Friction – friction through emission of *Cherenkov radiation* (examples: heavy atom moving through a Bose gas exhibiting BEC; dipole moving through optically dense medium; with Gang Zhou & Sof)
- Dynamics of quantum systems subjected to repeated measurements, emergence of facts in QM – statistics of measurement protocols, stochastic evolution eqs., large deviations; “theory of knowledge acquisition”.

Survey of examples, ctd.

(example: experiments of Haroche-Raimond group, electron conduction in presence of Coulomb blockade, etc.; with B, F, Sch)

- **Dynamics in limiting regimes** – mean-field limit, kinetic limit (Gross-Pitaevskii description of Bose gases, (Bogliubov-)Hatree-Fock: e.g., neutron stars; point-particle limit of NL Hartree Eq., etc.; with Schwarz, Kn. and others)

"Postmodern" examples of effective dynamics:

- Quantum chaos vs. quantum integr. behavior; e.v. statistics, …
- Quantum quenches, dynamical problems related to hard half of *0th Law of Thermodynamics* (e.g., "ETH")
- Many-body localization (see also results w. A; Sp & W; B & G)
- Very fast processes, such as ionization, involving (laser) light (F-P-S)
- Dynamics of inverted populations (dynamics of negative-temperature initial states; entanglement dynamics; …)

Unfortunately, I don't have much to say about these examples, yet – *who has?*

I will limit my attention to *quantum systems!*

A Remark on Transport and the Diffusion Constant

When trying to exhibit the (sub-)diffusive character of motion of a quantum particle hopping on a lattice $\mathbb{Z}^d$ one should study the following quantity: Let $\rho_t(x, y, \cdot)$ denote the one-particle density matrix at time t, with $\rho_0(x, y) = \delta_{x,0}\ \delta_{y,0}$. The "diffusion constant", D, is defined by

$$\sum_{x \in \mathbb{Z}^d} |x|^2 \mathbb{E}[\rho_t(x, x, \cdot)] \sim D \cdot t, \text{ as } t \to \infty, \tag{Diff-Const}$$

or, more accurately,

$$\sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{\sqrt{\tau}}\right) \mathbb{E}[\rho_{\tau t}(x, x, \cdot)] \underset{\tau \to \infty}{\sim} \frac{1}{(2\pi)^{d/2}} \int dx\ f(x) \exp\left[-\frac{x \cdot Dx}{2t}\right].$$

Here $\mathbb{E}$ denotes expectation w. r. t. a (transl.-invariant) "environment" influencing the motion of the particle; e.g., a heat bath of bosonic atoms, or phonons, or photons; or a random potential, as in the Anderson model; or it could be combination of such environments.

Heuristics

If

$$\sum_{x \in \mathbb{Z}^d} |x|^2 \mathbb{E}[\rho_t(x,x)]$$

grows like t^2 the motion is called ballistic, if this quantity grows more *slowly* than *linearly* in t then D vanishes; the motion is then called sub-diffusive. If it remains bounded in time t we say that the particle is localized. To get an idea under what circumstances one may expect *diffusive motion*, i.e.,

$$0 < D < \infty$$

we consider

$$\mathbb{E}[\rho([x(t) - x(0)]^2)] = \int_0^t \mathrm{d}r \int_0^t \mathrm{d}s \; \mathbb{E}[\rho(\dot{x}(r) \cdot \dot{x}(s))]. \qquad \text{(Vel-Cor)}$$

If the direction of motion of the particle de-correlates in time integrably fast, in the sense that $\mathbb{E}[\rho(\dot{x}(r) \cdot \dot{x}(s))]$ is intergrable in $|r - s|$, then $D < \infty$.

Results and Conjectures

If the environment consists of a translation-invariant heat bath of phonons or photons then

$$0 < D < \infty$$

provided the dimension $d > 3$ and the coupling of the particle to the heat bath is sufficiently weak (DeR & JF). If the environment consists of an ideal gas of non-relativistic bosonic atoms at positive temperature the same result holds if $d \geq 3$ (DeR & Kup).

If the environment is given by a static random potential $v_\omega(\cdot)$, where the values $\omega(x), x \in \mathbb{Z}^d$ of the potential are i.i.d. random variables (with, e.g., a bounded distribution of compact support), then the energy of the particle is a conserved quantity, and we may fix it in the expressions (Diff-Const) and (Vel-Cor). Then D depends on the energy E of the particle and is denoted by $D(E)$.

Results and Conjectures – ctd., Subject of Talk

Results:

- $D(E) = 0$, in d=1, $\forall E$.
- $D(E) = 0$ near the boundaries of the spectrum (band edges) of the Anderson Hamiltonian, or for sufficiently large disorder.
- Replace $v_\omega(\cdot)$ by $\lambda v_\omega(\cdot)$, with λ small. For $d \geq 3$, the motion is diffusive on time scales of order $\tau = \lambda^{-2-\varepsilon}$, ε small enough; (E-S-Yau).

Conjecture:

$$0 < D(E) < \infty,$$

in dimension $d \geq 3$, provided λ is small enough, and E is sufficiently far away from the band edges.

Subject of Talk

Study effect of thermal noise from a heat bath on motion of particle subject to a random potential; effects of repeated direct measurements.

2. Thermal noise kills Anderson localization

(with J. Schenker)

In this section we consider a quantum particle hopping on the simple cubic lattice $\mathbb{Z}^d$, d = 2,3, under the influence of a *random potential* and coupled to a *heat bath* at some positive temperature β^{-1}. The state of the particle is described by a one-particle density matrix,

$$\rho(x, y), x, y \in \mathbb{Z}^d. \qquad (1)$$

The dynamics of ρ is described by a Liouville equation

$$\partial_t \rho_t = \mathcal{L}(\rho_t), \qquad (2)$$

where t denotes time, and $\mathcal{L}$ is the 'Liouvillian' given by a *'Lindblad generator'* of the form

Thermal noise, ctd.

$$\mathcal{L} = -i \; ad_{H_\omega} + g(G - L), \;\; g > 0, \qquad (3)$$

where the Hamiltonian

$$H_\omega = -\Delta + v_\omega \qquad (4)$$

is a standard *random Schrödinger op.* (Anderson Hamiltonian), with $v_\omega = \{\omega(x)\}_{x \in \mathbb{Z}^d}$ a random potential (the $\omega(x)$ are iid rv's with, e.g., bd. distr., μ , of cpt. support), G is a *"gain term"* and L is a *"loss term"*. Introducing the variables $X = x + y$ and $\xi = x - y$, we can write the integral kernel of ρ as a function of X and ξ, i.e., as $\rho(X, \xi)$. Then G and L are given

Thermal noise, ctd.

by the formulae

$$(G\rho)(X,\xi) := \sum_{\eta\in\mathbb{Z}^d} r(\xi,\eta)\rho(X,\eta),$$

and

$$(L\rho)(X,\xi) := \sum_{\eta\in\mathbb{Z}^d} r(\xi-\eta,0)\rho(X,\eta),$$

(5)

where the kernel $r(\xi,\eta)$ satisfies a detailed-balance condition at infinite temperature.

Our main result is the following theorem.

Thermal noise, ctd.

Theorem. Consider Eq. (2) for ρ_t, with

$$\rho_{t=0}(x,y) = \delta_{x0}\delta_{y0}\,.$$

If the coupling constant $g > 0$ then the *diffusion constant*

$$D := \lim_{t\to\infty} t^{-1} \sum_x |x|^2 \mathbb{E}\rho_t(x,x), \qquad (6)$$

where $\mathbb{E}$ denotes an expectation w.r.t. to the product measure $\prod_{x\in\mathbb{Z}^d} \mu(\omega(x))$, exists and satisfies

$(i) \qquad 0 < D < \infty$

(ii) if the disorder is large enough for $D|_{g=0}$ to vanish (complete localization in absence of thermal noise) then

Thermal noise, ctd.

$$\lim_{g \searrow 0} \frac{D(g)}{g}$$

exists and is finite.

The *proof* of this theorem makes use of the following formalism. Let Ω denote the space of random variables $\omega(\cdot)$. We define the Hilbert space

$$\mathcal{H} := l_2(\mathbb{Z}^d \times \mathbb{Z}^d) \otimes L^2(\Omega, \Pi_x \mu(\omega(x)))$$

Fourier transformation of vectors, Ψ, in $\mathcal{H}$ is defined by

$$\widehat{\Psi}(k, x; \omega) := \sum_{a \in \mathbb{Z}^d} e^{-ik\cdot a} \Psi(x + a, a; \tau_a \omega)$$

Let $\hat{\rho}_t(k, x; \omega)$ denote the Fourier transform of ρ_t.

Thermal noise, ctd.

Then $\hat{\rho}_t(k, x; \omega)$ satisfies the equation

$$\partial_t \hat{\rho}_t(k) = -\mathcal{G}_k \hat{\rho}_t(k),$$

where $\mathcal{G}_k$ = "Fourier transform of $\mathcal{L}$" (see blackboard).

Then the diffusion constant D/d is given by the expression

$$-\lim_{t\to\infty}\frac{1}{t}\partial_{k_1}^2 \sum_x e^{-ik\cdot x}\mathbb{E}\rho_t(x,x)|_{k=0} = -\lim_{t\to\infty}\frac{1}{t}\partial_{k_1}^2 \mathbb{E}\hat{\rho}_t(k,0;\cdot)|_{k=0}.$$

Defining the vector $\phi := (\delta_{e_1} - \delta_{-e_1}) \otimes \mathbf{1}$, we find that

$$D/d \;=\; \lim_{\epsilon\searrow 0}\langle\phi, \frac{1}{\mathcal{G}_0+\epsilon}\phi\rangle$$

The R.S. can be estimated using the Feshbach-Schur map.

Well, this is a little sketchy! (But details are fairly easy.)

Some details

1. Assumptions on the kernel $r(\xi,\eta)$: pg. 5! (r of positive type, r is kernel of a bounded operator on $\ell_2(\mathcal{Z}^d)$, $L - G$ positive, ...)
2. Expression for $\mathcal{G}_k$:

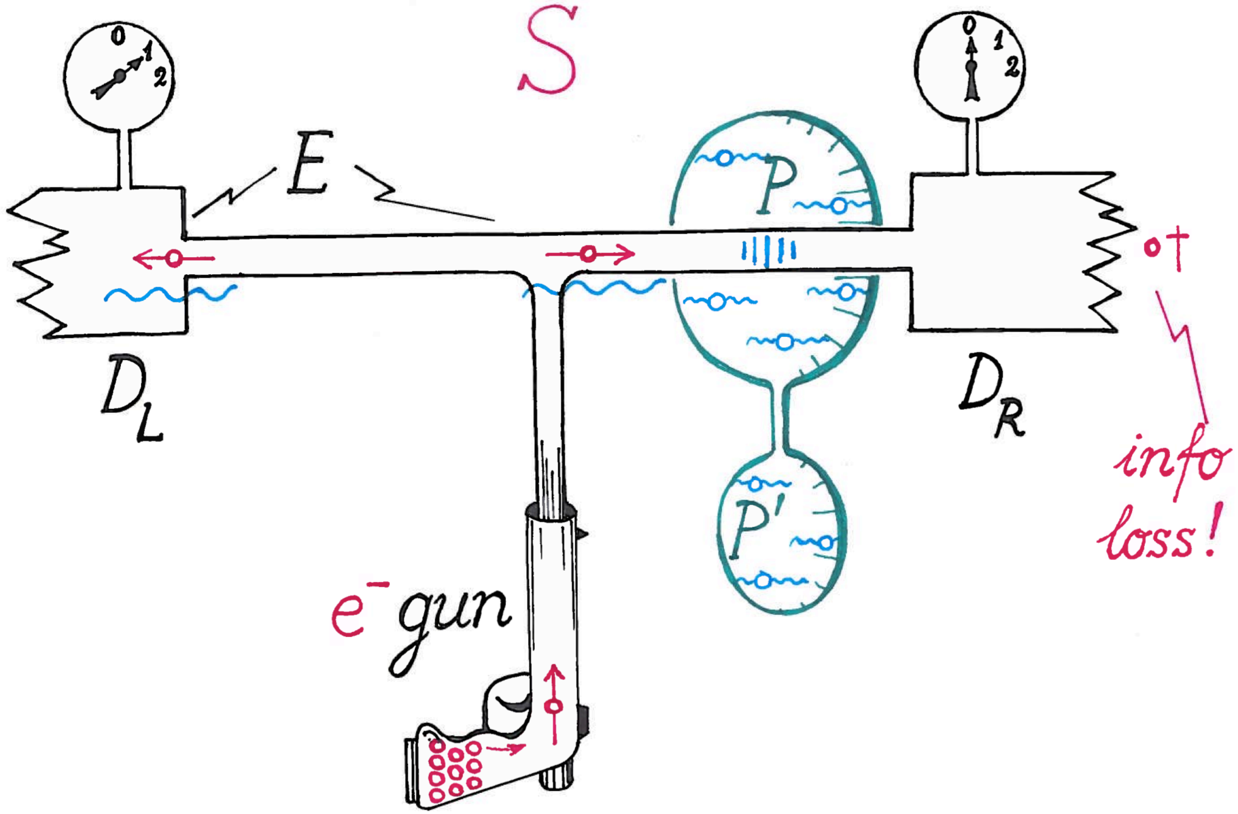
$$\begin{aligned}(\mathcal{G}_k\hat{\rho})(k,x,\omega) &= \sum_{|e|=1}[\hat{\rho}(k,x+e,\omega) - e^{-ik\cdot e}\hat{\rho}(k,x+e,\tau_e\omega)] \\ &\quad +(\omega(x)-\omega(0))\hat{\rho}(k,x,\omega) \\ &\quad +\sum_y e^{ik\cdot\frac{y-x}{2}}[r(x,y)-r(0,y-x)]\hat{\rho}(k,y,\tau_{\frac{y-x}{2}}\omega).\end{aligned}$$

3. Expression for diffusion constant D:

$$\begin{aligned}\frac{1}{d}\sum_x |x|^2\mathbb{E}\hat{\rho}_t(x,x,\omega) &= -\partial_{k_1}^2\mathbb{E}\hat{\rho}_t(k,0,\omega) \\ &= -\partial_{k_1}^2\langle\delta_0\otimes 1, e^{-t\mathcal{G}_k}\delta_0\otimes 1\rangle_{\mathcal{H}}|_{k=0}\end{aligned}$$

$\Rightarrow$ expression for D! (Pass to (II.5), pg. 15, and pg. 16.)

3. Dynamics of a Quantum System under repeated measurements – "theory of knowledge acquisition"



Main features of model

$$S = (P \vee P') \vee E$$

At all times, only *one* e^- in “T-channel” $\subset E$.

The only possible projective measnt. in S is to observe whether D_L or D_R has clicked. This measurement is represented by an operator, X, given by

$$X = \mathbf{1}_{\overline{P}} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_E \qquad (6)$$

Question: What can one learn about S, in particular about P, by performing long sequences of successive measnts. of X ? Random outcomes of measnts. of X ➔ *eff. stoch. dynamics!*

Main features of model, ctd.

Up to N e^-' s bound by P create "Coulomb blockade" in arm of "T" reaching to D_R, thus discouraging e^- moving inside "T" to be scattered onto D_R. Simple qm calculations yield the probabilities, $p_L(n)$ and $p_R(n)$, for an e^- to be scattered onto D_L, D_R, respectively. Here n is the number of electrons bound by P. Clearly

$$p_L(n) + p_R(n) = 1. \qquad (7)$$

Note that n is an eigenvalue of the electron number operator $\mathcal{N}$ whose eigenvalues correspond to the number of electrons bound by P.

We now imagine that every τ seconds an e^- is injected into "T" and gets scattered onto one of the two detectors, thus resulting in a *measurement* of X. Let $\xi = \pm 1$ denote the eigenvalues of X (corresp. to a click of D_L, D_R, resp.)

Main features of model, ctd.

A *measurement protocol* of length k consists of outcomes

$$\underline{\xi}^{(k)} = (\xi_1, ..., \xi_k), \ k = 1, 2, 3, ... \qquad (8)$$

of first k measurements of X. Choosing a state ρ of S enables one to associate a *probability* (or "frequency")

$$\mu_\rho(\xi_1, ..., \xi_k) \qquad (9)$$

with each measurement protocol, $\underline{\xi}^{(k)}$, of length k =1,2,3,...

(The unique qm formula for the probabilities (9) has first been found by Schwinger and rediscovered by Wigner,...)

One has that

$$\sum_{\xi_k} \mu_\rho(\xi_1, ..., \xi_{k-1}, \xi_k) = \mu_\rho(\xi_1, ..., \xi_{k-1})$$

Let Ξ denote the space of all ∞ long measurement protocols. Then μ_ρ defines a probability measure on Ξ.

(If μ_ρ is inv. under permutations then use *De Finetti*!)

4. Indirect measurements & pointer observables

We define the *frequency of clicks* of D_L in first k measnts. by

$$\nu_L(\underline{\xi}^{(k)}) := \frac{1}{k}\sharp\{j \in \{1, \dots, k\} | \xi_j = 1\} \tag{10}$$

and $\nu_L(\underline{\xi})$ the limit of $\nu_L(\underline{\xi}^{(k)})$, as $k \to \infty$. We expect:

$$\nu_L(\underline{\xi}) = p_L(n), \text{ where } n \text{ is an eigenvalue} \tag{11}$$

of the number operator $\mathcal{N}$ introduced after (7).

Important observation:

There is a close analogy between *S*, with (7) through (11), and the *classical stat. mechanics of Spin Chains (SC)*:

time of *S* $\leftrightarrow$ *1D space* of SC; $\mu_\rho \leftrightarrow$ Gibbs state of SC

Indirect measurements, ctd.

Limit, as (discrete) time $k \to \infty \leftrightarrow$ *TD limit* of SC

Appealing to *"equivalence of ensembles"* in TD limit, we expect that the *fluctuations* of $\nu_L(\underline{\xi}^{(k)})$ around one of the possible limiting values $p_L(n)$ tend to 0, as $k \to \infty$.

Precise statement

We define

$$\Xi_n(k;\underline{\epsilon}) := \{\underline{\xi} \in \Xi \mid |\nu_L(\underline{\xi}^{(k)}) - p_L(n)| < \epsilon_k\}$$

and (12)

$$\Xi(k;\underline{\epsilon}) := \cup_{n=1}^{N} \Xi_n(k;\underline{\epsilon}) \subset \Xi$$

where $\epsilon_k \searrow 0$, as $k \to \infty$.

Indirect measurements, ctd.

Theorem.

Hyp. *(ND measnt.)*: Operator $\mathcal{N}$ $(t) = \mathcal{N}$ const. in time t;

& suppose that $\Delta := min_{n_1 \neq n_2} \mid p_L(n_1) - p_L(n_2) \mid > 0$.

Then:

(A) If k is so large that $\epsilon_k < \Delta/2$ then the sets

$$\Xi_1(k, \underline{\epsilon}), \ldots, \Xi_N(k, \underline{\epsilon})$$

are all disjoint from one another; and

(B) $$\mu_\rho(\Xi(k; \underline{\epsilon})^c) = 1 - \mu_\rho(\Xi(k; \underline{\epsilon})) < \delta_k,$$

with $\delta_k \searrow 0$, as $k \to \infty$.

The sequences $\underline{\epsilon}$ and $\underline{\delta}$ can "usually" be chosen to be independent of the state ρ; (appropriate hyp.!)

<u>*General insight*</u>

Think of a more general quantum system, *S*.

5.Analogy to classical statistical mechanics

Imagine that many successive projective measurements of X at times $t_1 < t_2 < \cdots < t_k$ are made, with outcomes $\underline{\xi}^{(k)} = (\xi_1, \dots, \xi_k)$, where $\xi_j \in \mathrm{spec}(X)$ is the value of X measured at time t_j, $j = 1,\dots, k$, $k = 1,2,3,\dots$.

Given any state ρ of S, one may predict the *emp. prob.*, $\mu_\rho(\xi_1, \dots, \xi_k)$, of the measnt. protocol $\underline{\xi}^{(k)} = (\xi_1, \dots, \xi_k)$.

Let $\Xi_S \subset \Xi$ be the smallest set of arbitrarily long measnt. prot. s.t. $\mu_\rho(\Xi_S^c) = 0$, for all states, ρ, of S.

We define $\mathcal{O}_{S,\infty}$ to be the algebra of functions on Ξ_S *measureable at* ∞, which are also called *"pointer obs."*.

(If S is autonomous and if there is complete decoherence then $\mathcal{O}_{S,\infty}$ consists of *all fus. invariant under right-shift*!)

Analogy to class. Stat. mech., ctd.

An example of a pointer observable is $\nu_L(\underline{\xi})$; (see Eq. (10)). The projections (characteristic functions) in the algebra $\mathcal{O}_{S,\infty}$ describe *events* in the system *S* detected with the help of very many successive measurements of *X.* *All states* ρ */ probability measures* μ_ρ *can be decomposed into a convex combination of mutually singular states/ prob. measures indexed by points in the spectrum of* $\mathcal{O}_{S,\infty}$ $(= \mathcal{E}_\infty := \bigcap_t \mathcal{E}_{\geq t}$); disjoint supports!

In our simple model, $\mathcal{O}_{S,\infty}$ consists of all functions on {1,…,N}, which is the spectrum of the number operator $\mathcal{N}$.

Many successive measurements of *X* (detector clicks) provide information about number of e^- bound by dot *P.*

6. Effective dynamics of pointer observables

But most indirect measurements are *not* non-demolition measurements. Usually $\mathcal{O}_{S,\infty}$ is *empty*. However, there are functions, $\vec{\nu}_k$, on Ξ_S depending on measurement protocols

$$\underline{\xi}^{(m,k)} = (\xi_{mk+1}, \ldots, \xi_{(m+1)k}), \;\; m = 0, 1, 2, \ldots$$

of length *k*, which approximate *pointer observables* – example: $\nu_L(\underline{\xi}^{(m,k)})$ – whose level sets provide a decomposition of Ξ_S into disjoint subsets,

$$\Xi_{\underline{\sigma}}(m, k; \underline{\epsilon}), \;\; \underline{\sigma} \in \Sigma, \qquad (13)$$

with the property that the μ_ρ - measure of the complement of their union, $\bigcup_{\underline{\sigma} \in \Sigma} \Xi_{\underline{\sigma}}(m, k; \underline{\epsilon})$, is tiny.

Effective dynamics, ctd.

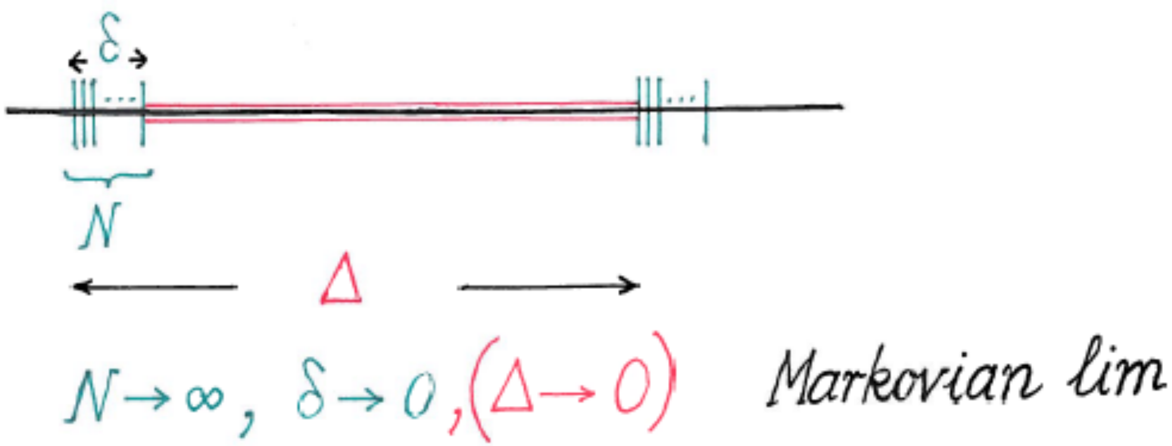
A measurement protocol of length *Mk*, (*M*$<\infty$ not too large) then determines a *trajectory* (or *"history"*)

$$\{\underline{\sigma}(m)\}_{m=1,2,3,\ldots} \tag{14}$$

of *events*, $\underline{\sigma}(m)$, in Σ, with $m = 1,2,\ldots,M$.

<u>Important problem</u>

We would like to determine the *effective stochastic dynamics of event histories* ("eth"). Here is a very simple model, where this can be done!



Effective dynamics, ctd.

In this model, the state tends to purify after $N \gg 1$ projective measurements of X within a time interval of length δ and then evolves *unitarily* during a time interval of length $\Delta - \delta$. In the limiting regime, where first N tends to ∞ and then δ to 0, a *Markov chain* with transition function

$$p(n, n') := |[exp(-i\Delta H)]_{n,n'}|^2 \qquad (15)$$

describes the *effective dynamics of event histories* consisting of a jump process on the spectrum of the number operator $\mathcal{N}$.

There is a more interesting limiting regime that can be analyzed with the help of Kurtz' stochastic Trotter product formula. But we don't have time to describe it here.

7. Conclusions

"In all my films, I have been faithful to these suspension points in the conclusions. Besides, I have never written the word 'END' on the screen."
(Federico Fellini)

"Everyone wants to understand art (physics)*. Why don't we try to understand the song of a bird? Why do we love the night, the flowers, everything around us, without trying to understand them? But in the case of a painting* (result in physics)*, people think they have to understand." (Pablo Picasso)*

Thank you for listening!